A Hierarchical Re-weighted-ℓ_1 Approach for Dynamic Sparse Signal Estimation

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Compressive sensing results have allowed accurate reconstruction of highly undersampled signals by leveraging known signal structure [1]. Recently, there has been a push to extend these results into an area of great interest for a large number of fields: the estimation of dynamically changing signals [2]–[5]. If known, or even partially known dynamics are transforming a state, then past observations should be able to be incorporated into the estimation process of a state at any given time in order to increase the accuracy of the estimation. Typically a dynamical state $\boldsymbol{x}_n \in \mathbb{R}^N$ is assumed to evolve with some approximately known dynamics $f_n(\cdot)$ as

$$\boldsymbol{x}_n = f_n(\boldsymbol{x}_{n-1}) + \boldsymbol{\nu}_n, \tag{1}$$

where $\boldsymbol{\nu}_n$ is called the innovations and can be interpreted as the limitation of our knowledge of the system dynamics. Given a set of linear measurements at each iteration,

$$\boldsymbol{y}_n = \boldsymbol{G}_n\boldsymbol{x}_n + \boldsymbol{\epsilon}_n, \tag{2}$$

where $\boldsymbol{y}_n, \boldsymbol{\epsilon}_n \in \mathbb{R}^M$ are the measurements and measurement error, respectively, we wish to estimate the underlying evolving state. More specifically, we wish to recover the current state at each time step as best as possible given all previous measurements. In previous work [4], we explore a framework in which propagating first order statistics and utilization of appropriate ℓ_1 norms allow for accurate estimation when the state, the innovations or both are sparse.

In least-squares based state estimation, however, higher order statistics are propagated in order to obtain more accurate estimates at each iteration. For instance in the case of the Kalman filter, which arises when under assumptions of linearity in the modeled dynamics and Gaussian statistics in the innovations and measurement noise, a covariance matrix is propagated along with the mean to obtain an optimal estimate. In this work, we expand on the previously introduced framework in order to include similar higher order statistics by introducing a hierarchical model inspired by the reweighted ℓ_1 sparse inference method first proposed in [6]. We use previous information in a way similar to [7] in that we are leveraging the weightings $\boldsymbol{\Lambda} = diag(\lambda_i)$ in the optimization

$$\hat{\boldsymbol{x}} = \arg\max_{\boldsymbol{x}} \|\boldsymbol{y} - \boldsymbol{G}\boldsymbol{x}\|_2^2 + \|\boldsymbol{\Lambda}\boldsymbol{x}\|_1 \tag{3}$$

in order to propagate information about our prediction and our confidence thereof of the next state. By using a Gamma prior over each element of $\boldsymbol{\lambda}$ in a Bayesian setting, we determine the expectation-maximization (EM) update equations in order to determine $\boldsymbol{x}_n$ and $\boldsymbol{\lambda}_n$ at each iteration to be

$$\boldsymbol{\lambda}^t[i] = \frac{2}{|\boldsymbol{x}^{t-1}[i]| + f_n(\boldsymbol{x}_{n-1})[i] + \beta} \tag{4}$$

$$\boldsymbol{x}_n^t = \arg\min_{\boldsymbol{x}} \left[\|\boldsymbol{y}_n - \boldsymbol{G}_n\boldsymbol{x}\|_2^2 + \sum_i \boldsymbol{\lambda}^t[i]|\boldsymbol{x}[i]| \right] \tag{5}$$

where β is a small positive value which ensures stability in the $\boldsymbol{\lambda}$ values and t indicates the EM iteration. The EM algorithm run to convergence, which typically occurs for $10 \leq t \leq 30$.

We show improvements on simulated data using the adaptation of the second order variables over similar first order estimation programs in both the steady state relative mean squared error (rMSE) and the robustness. For example at sampling rates below CS recovery limits, steady state errors can be reduced from 2.48% using first order methods to 0.67% with the re-weighted model. Additionally, up to 30% of the signal sparsity locations can be erroneous and the re-weighted model continues to outperform both time-independent basis pursuit de-noising as well as the first order models.

References

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